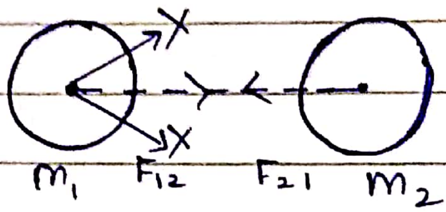
Physics - Gravitation -

*.) Newton's law of gravitation -

$F \propto m_1 m_2$ and $F \propto \frac{1}{r^2}$

$\therefore F = \frac{G M_1 M_2}{r^2}$ where G = Universal gravitational constant.
$\{G = 6.67 \times 10^{-11} Nm^2/kg^2\}$

i.e,

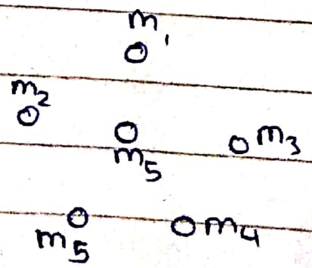


- The force is central.
- The force is Attractive
- The force is independant of medium.
- $\vec{F_{12}} = -\vec{F_{21}}$
- Valid Only for point masses.
- Significant only for Heavy bodies
- Gravitation is an Internal Force.

*.) Principal of Superposition :-

If no. of forces are acting on a point mass, then the net force on point is the vector addⁿ of all the forces.

i.e :-



$\{ F_{net}(m_n) = F_{nm_1} + F_{nm_2} + F_{nm_3} \ldots F_{n(n-1)} \}$

Date

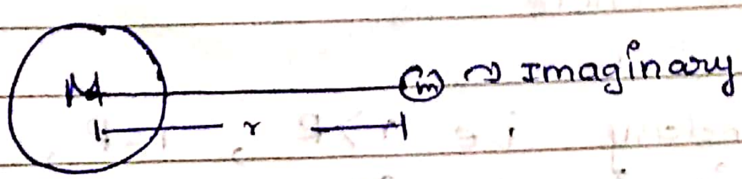
*) Gravitational Field :-

Gravitational field Strength / Intensity ($\vec{E}$) ,

i.e $\vec{E} = \dfrac{\vec{F}}{m}$, unit = N/kg or m/s^2.

for multiple GFS :- $\{E_n = E_1 + E_2 \ldots\ldots\}$

• Gravitational field due to Has point mass 'M' —



$\because F_{mM} = \dfrac{GmM}{r^2}$

$\because \vec{E} = \dfrac{F}{m} = \dfrac{G\cancel{m}M}{r^2\cancel{m}} = \left\{-\dfrac{GM}{r^2}\right\}$.

• To find force at a point mass :-

$F = mE$ where E is Gravitational feild strength.

• General formula to Calculate gravitational field at any angle pla for a point mass :-

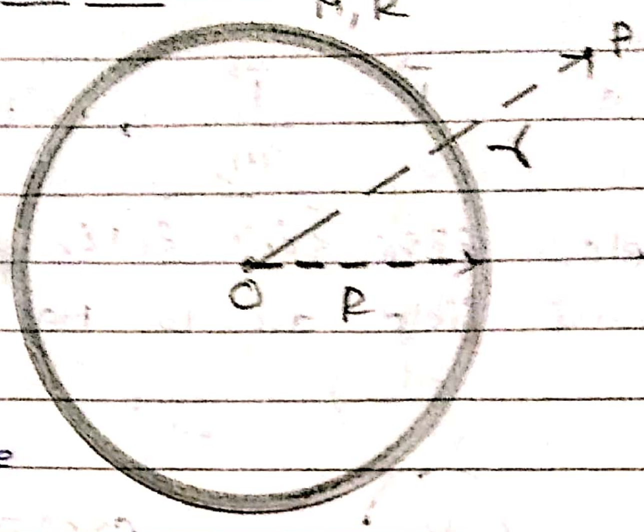
$$\left\{ E = \frac{GM}{R^2\phi} \, \frac{2\sin\phi}{2} \right\}$$

*.) Gravitational field due to Hollow + Solid Sphere.

1.) **For Hollow Sphere :-**

mass = M.

radius = R



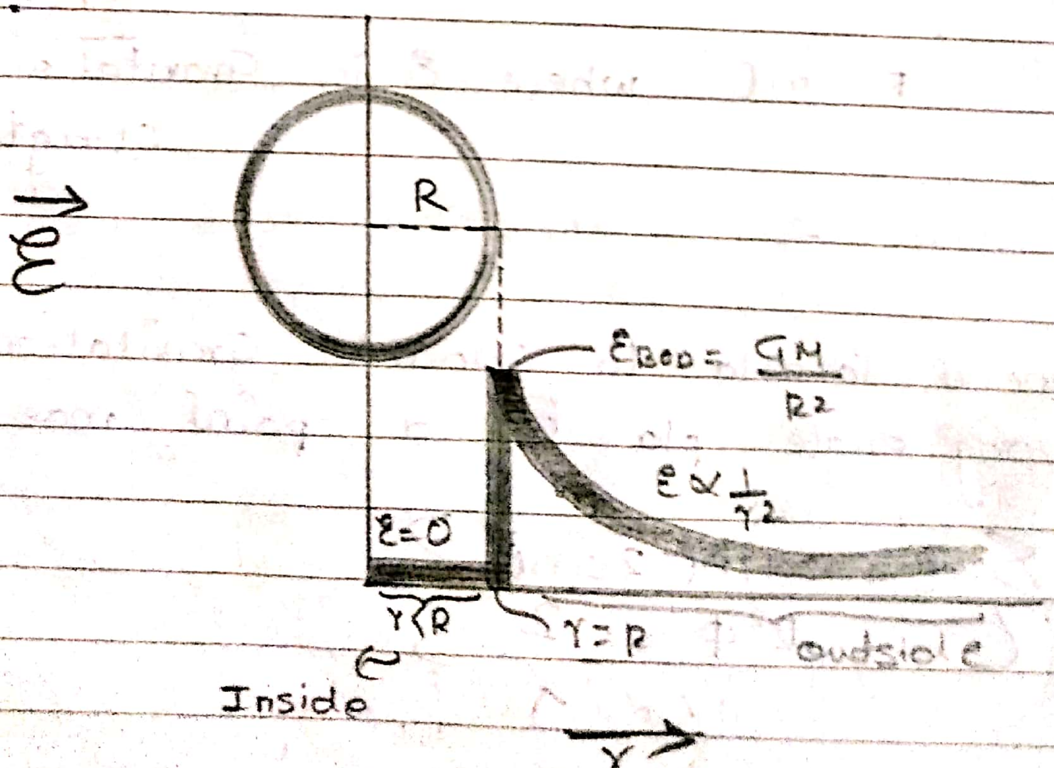
Three points are chosen

- Out side circle
- Inside circle
- At boundary i.e $r > R$, $r = R$, $r < R$

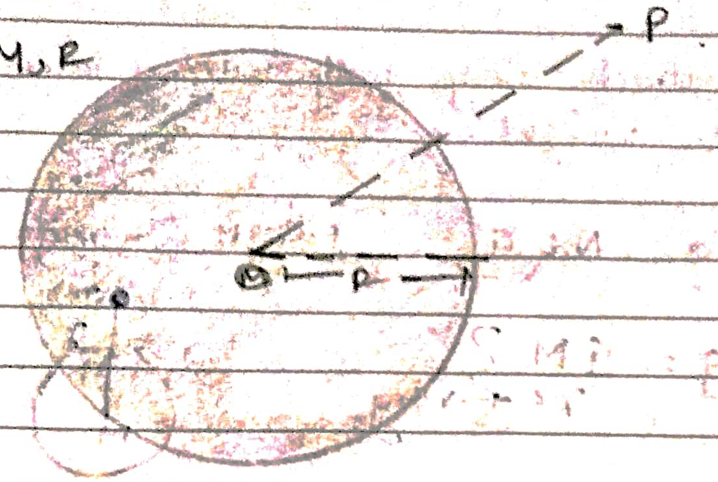
$\therefore$ Gravitational field,

- Outside :- $E_{out} = \frac{GM}{r^2}$, • E_{BOD} :- $E = \frac{GM}{R^2}$
- Inside, i.e $r < R$, $E = 0$ [Gauss law of gravitation]

Graph:-



2.) For Solid Cylinder -

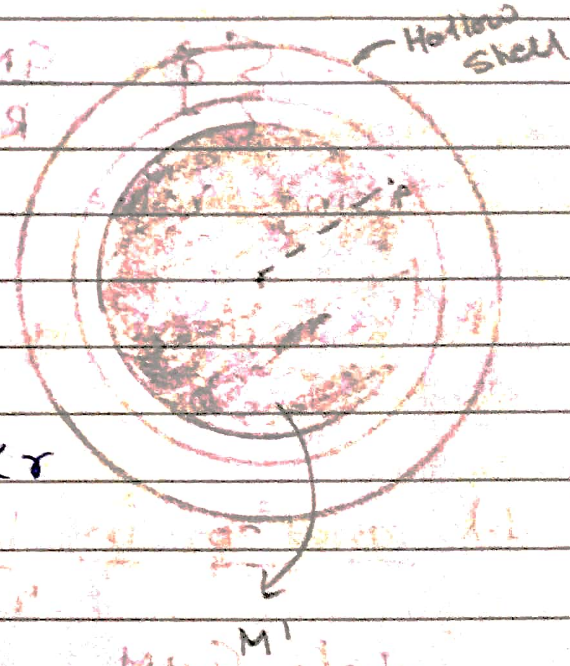


Outside $E = \frac{GM}{r^2}$, $E \propto \frac{1}{r^2}$

At Boundary $E = \frac{GM}{R^2}$ {constant}

Inside :- $E_{Hollow} + E_{M'}$

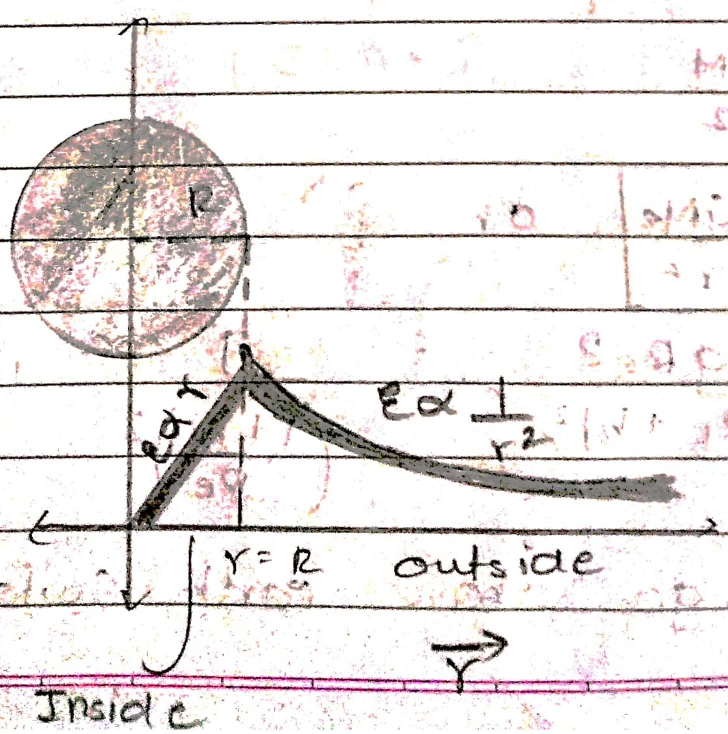
$= 0 + E_{M'}$



for M', Point P is on Boundary

$\therefore E_{M'} = \frac{GMr}{r^3} = \frac{4G\rho\pi r}{3}$, $E \propto r$

Graph :-



for earth $\vec{E} = \vec{g}$

$G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

$M = 6 \times 10^{24}\ kg.$

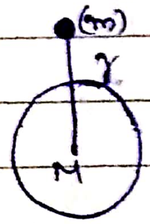
$R = 6400\ km = 6.4 \times 10^{6}\ m$

*) Acceleration due to gravity :- $(\vec{g}) = 9.8\ m/s^2 \approx 10\ m/s^2$

Earth attracts every body with a force equal to $= m \times g$ called weight.

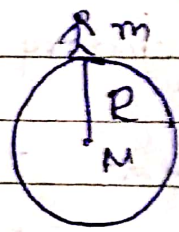
applying N.L.G, $F = \frac{G m M}{r^2} = mg$

$$\text{vary}\left\{ g = \frac{GM}{\underbrace{r^2}_{\text{vary}}} \right\}$$



The value of $\vec{g} = 9.8 / 10\ m/s^2$ is taken at surface of earth.

i.e $r = R$



$$\therefore \left\{ \vec{g} = \frac{GM}{R^2} \right\}$$

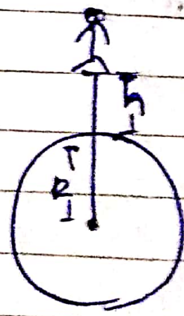
using values, we get $\vec{g} = 9.8\ m/s^2$ or $10\ m/s^2$

*) Variation in $\vec{g}$:-

1.) Variation in height / Depth from Surface :- $(\vec{g})$:-

$mg' = \frac{G m M}{r^2}$

$$\boxed{g' = \frac{GM_e}{r^2}} \quad \text{or}$$



$$g' = \frac{g R_e^2}{(R_e + h)^2} = \frac{g}{\left(1 + \frac{h}{R_e}\right)^2}$$

As we go above earth surface, $r \uparrow$, $g' \downarrow$

$$\left\{ \therefore g' = \frac{g}{\left(1+\frac{h}{R_e}\right)^2} \right\}$$

• Special Case :-

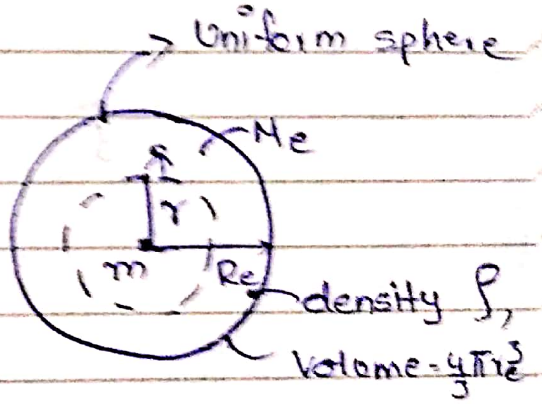
If $h << R_e$, then,

$$\left\{ g' = g\left(1+\frac{h_e}{R_e}\right)^{-2} = g\left(1-\frac{2h}{R_e}\right) \right\} \because (1+x)^{-n} (= 1-nx).$$

*) variation in 'g' as we go Below Earth's Surface :-

$$mg' = \frac{GmM'}{r^2}$$

$$\therefore \left\{ g' = \frac{GM'}{r^2} \right\}$$



$$\therefore g' = \frac{G \times \rho \times \frac{4}{3}\pi r^3}{r^2}$$

$$\therefore \left\{ g' = \frac{4}{3} G\rho\pi r \right\}$$

again

$$g' = \frac{4}{3} G \times \frac{M_e}{\frac{4}{3}\pi R_e^3} \times \frac{4}{3}\pi r$$

$$= \left\{ g' = \frac{G M_e r}{R_e^3} \right\}$$

again,

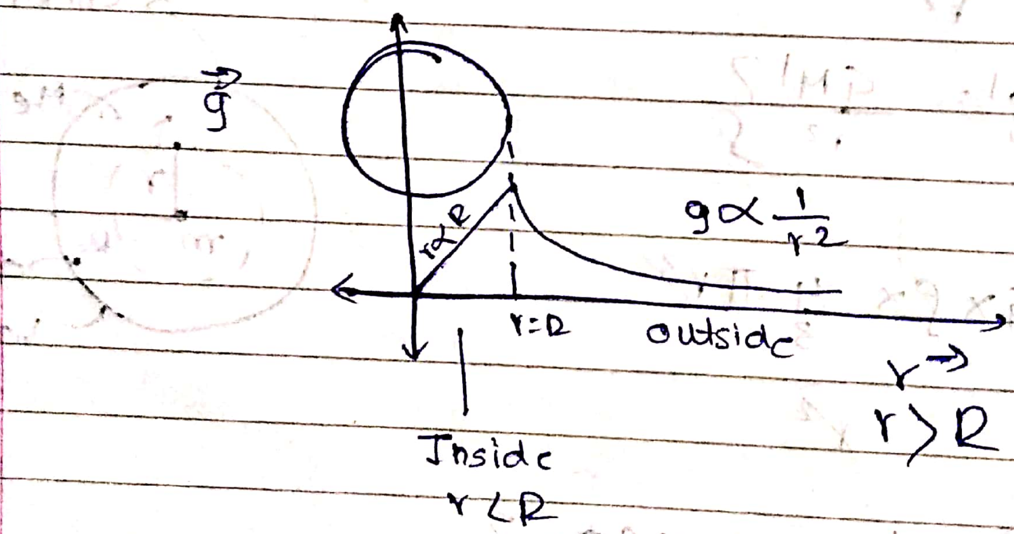
$$g' = \frac{GM_e r}{R_e^3}$$

$$g' = \frac{gR_e^2 r}{R_e^3} \qquad \left[\because g = \frac{GM_e}{R_e^2}\right]$$

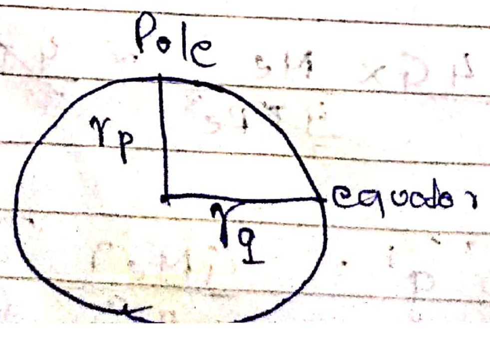
$$\therefore g' = \left(\frac{g}{R_e}\right) r$$

$$\left\{ \therefore g' = g\left(1 - \frac{h}{R_e}\right) \right\} \because (r = R_e - h)$$

(JEE main) :-



Variation in 'g' due to Shape of Earth



$r_p < r_q$

or

$r_q > r_p$

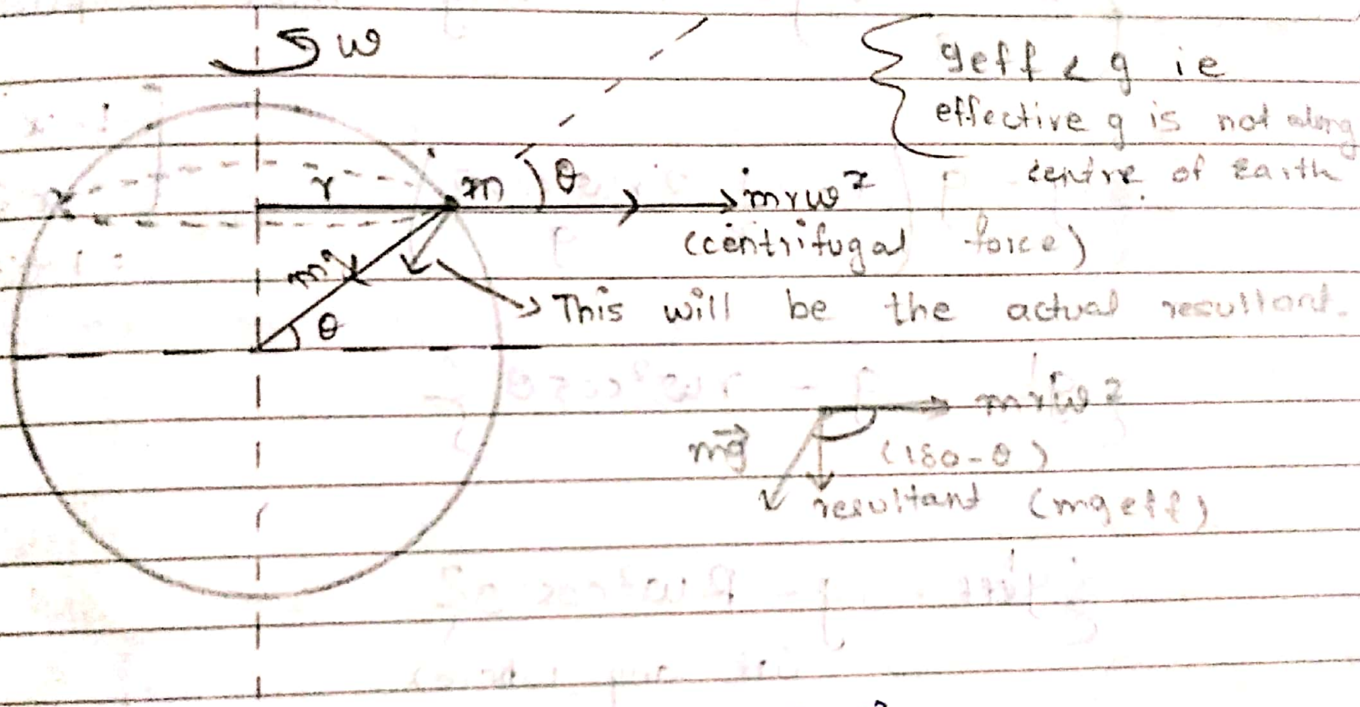
$\therefore \{g_{eq} < g_{pole}\}$ (experimentally proved).

The results were true experimentally but did not confirm with calculation.

The problem was with the Concept of 'Shape' So they introduced a new concept below.

*) Variation of '$\vec{g}$' due to Rotation of Earth :-

θ) latitude



$\therefore \; mg_{eff} = mg + mr\omega^2$

$\therefore \{\vec{g}_{eff} = \vec{g} + \vec{r}\omega^2\}$

Using Vector Add^n property, resultant will be

$\therefore g_{eff} = \sqrt{(g)^2 + (r\omega^2)^2 + 2(g)(r\omega^2)\cos(180-\theta)}$

$g_{eff} = \sqrt{g^2 + r\omega^4 - 2gr\omega^2\cos\theta}$

$\because \omega \rightarrow$ very small $\therefore r\omega^4 \rightarrow 0$

$\therefore$ it's neglected.

$\therefore g_{eff} = \sqrt{g^2 - 2gr\omega^2\cos\theta}$

$= g\sqrt{1 - \frac{2\cancel{g}r\omega^2\cos\theta}{g}}$

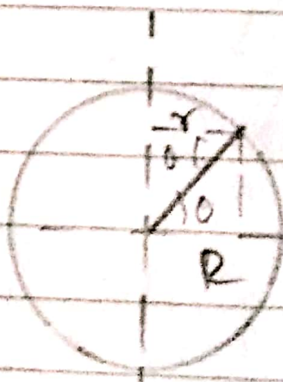
$\therefore \quad g' = g\left(1 - \frac{2r\omega^2\cos\theta}{g}\right)^{\frac{1}{2}}$

$\left(\frac{2r\omega^2\cos\theta}{g}\right)^{\frac{1}{2}}$ is very small compared to 1

$[1-x]^n$
$x << 1$
$= 1 - nx$

$\therefore g' = g\left(1 - \frac{1}{\cancel{2}}\frac{\cancel{2}r\omega^2\cos\theta}{g}\right)$

$\{g' = g - r\omega^2\cos\theta\}$



$r = R\cos\theta$

$\{g'_{eff} = g - R\omega^2\cos^2\theta\}$
(at any where)

- At equator, $\theta = 0$
  $\therefore \{g'_{eff} = g - R\omega^2\} \rightsquigarrow$ min.

- At pole, $\theta = 90$
  $\therefore \{g' = g\} \rightsquigarrow$ max.

*) Gravitational Potential :-

Potential Difference :-

$$\left\{ V_A - V_B = \frac{W_{B \to A}}{m} \right\}$$

$A \xleftarrow{W} B$ (m)

[with no change in KE i.e $\Delta KE = 0$ of mass.]

Potential at a point :- [We take a frame of reference at ∞] i.e $V_\infty = 0$.

$A \xleftarrow{W} B\ \infty$ (m)   [K.E = 0, slow, $\Delta KE = 0$]

$$\therefore V_A - V_\infty = \frac{W_{\infty \to A}}{m}$$

$$\therefore V_A = \frac{W_{\infty \to A}}{m}$$

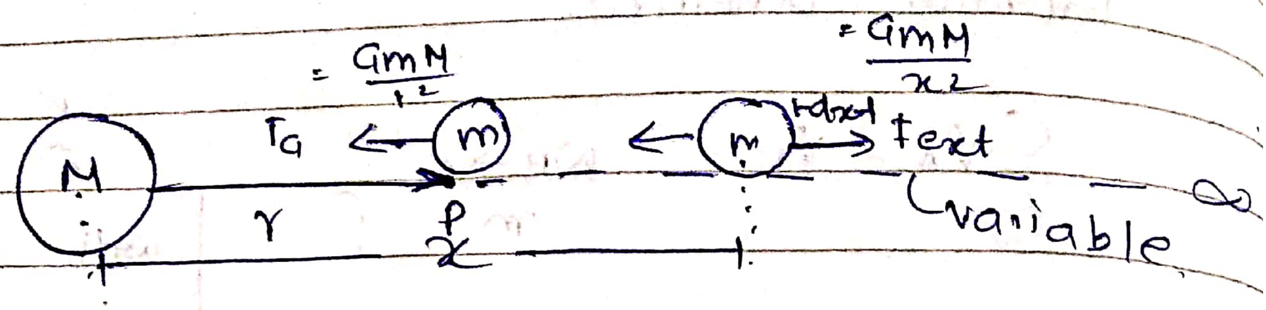
*) Gravitational potential due to a mass 'M' :-

$M \xrightarrow{\ \vec{r}\ } P \xleftarrow{W} m\ \infty$ (F_G ←, → F_{ext})

$$V_P = \frac{W_{\infty \to P}}{m}$$ → work done by external agent.

Scalar Quantity ← $\left\{ V_P = \frac{-GM}{r} \right\}$

Derivation :-



$$\underset{P \to \infty}{W_{ext}} = \int F \cdot dx = \int_P^{\infty} F \cdot dx = \int_r^{\infty} \frac{GmM}{x^2}\, dx$$

$$= GmM \int_r^{\infty} \frac{dx}{x^2} = GmM \left[-\frac{1}{x} \right]_r^{\infty}$$

$$= GMm \left[-\frac{1}{\infty} - \left(-\frac{1}{r} \right) \right] = \left\{ \frac{GMm}{r} \right\} \quad \left[\because \frac{1}{\infty} \right.$$

$$\therefore \text{G.P. at } P = \frac{W_{\infty \to P}}{m} = \frac{GM\cancel{m}}{r\cancel{m}}$$

$$\boxed{\therefore \text{G.P at } P = -\frac{GM}{r} \text{ due to Mass } M}$$

or

$$\boxed{\begin{aligned} W_{A \to B} &= m(V_B - V_A) \\ W_{A \to B} &= m\Delta V \end{aligned}}$$

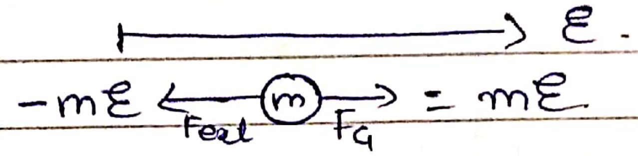
*) Relationship b/w Gravitational field & Gravitational Potential :- ($\vec{E}$ & V) :-

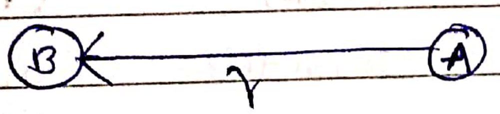
$$\left\{ \vec{E} = \frac{-dV}{dr} \right\}$$

In particular direction,

$$\left\{ E_x = \frac{-\delta V}{\delta x} \quad , \quad E_y = \frac{-\partial V}{\partial y} \quad , \quad E_z = \frac{-\delta V}{\delta z} \right\}$$

Derivation-





$W_{ext\,A-B}$ = F x disp.

= $-m\vec{E}r$

$\therefore V_B - V_A = \frac{W_{A-B}}{m} = \frac{-\cancel{m}Er}{\cancel{m}}$

$\therefore V_B - V_B = -Er.$

If, Distance b/w A & B is small, in potential Difference will also be small,

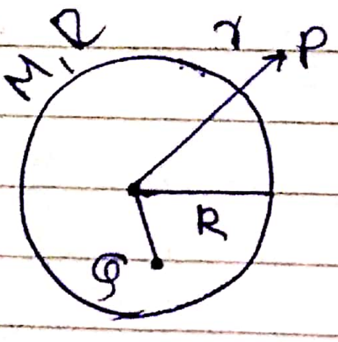
$\therefore dV = -E\,dr$

$$\therefore \left\{ E = \frac{-dV}{dr} \right\}$$

*·) Gravitational Potential And Field :-

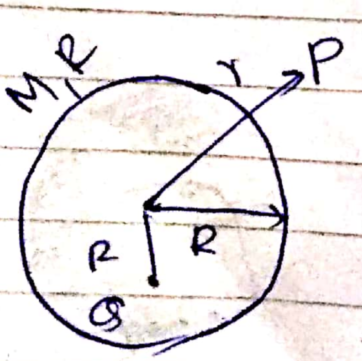
1.) For Hollow Sphere :-

Or Shell :-



Hollow Sphere	Outside $(r>R)$	Surface $(R=r)$	Inside $(r<R)$
$\vec{E}$:-	$\frac{-GM}{r^2}$	$\frac{-GM}{R^2}$	0
V :-	$-\frac{GM}{r}$	$-\frac{GM}{R}$	$\frac{-GM}{R}$

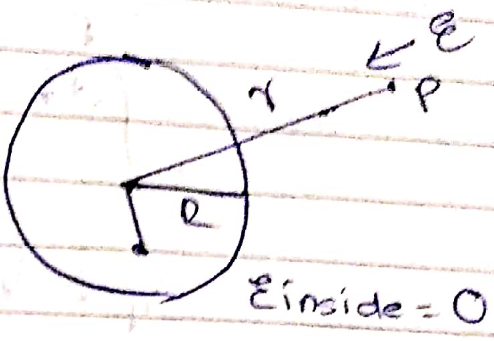
2.) For Solid Spher :-



	$(r>R)$ Outside	$(r=R)$ Surface	$(r<R)$ Inside
$\vec{E}$:-	$-\frac{GM}{r^2}$	$-\frac{GM}{R^2}$	$-\frac{GMr}{R^3}$
V :-	$-\frac{GM}{r}$	$\frac{-GM}{R}$	$-\frac{GM}{R^3}\left[\frac{3R^2}{2}-\frac{r^2}{2}\right]$

★ Derivations —

1) For Hollow Sphere –



- Gravitational field at P :- $\frac{-GM}{r^2}$.

and, $\mathcal{E} = -\frac{dV}{dr}$

For gravitational Potential,

$$\int_{\infty}^{P} dV = \int_{\infty}^{P} -\mathcal{E}\,dr$$

$$= V_P - V_\infty = \int_{\infty}^{r} \frac{GM}{r^2}\,dr$$

$$\therefore V_P - V_\infty = GM\left[\frac{-1}{r}\right]_{\infty}^{r} = GM\left[\frac{-1}{r} - \left(\cancel{\frac{-1}{\infty}}\right)\right]$$

$$\therefore \left\{ V_P = -\frac{GM}{r} \right\} \ldots\ldots \text{(for outside)}$$

- at Surface, $r = R$.

∴ G. Potential at surface

$$= \left\{ V_s = \frac{-GM}{R} \right\} \ldots\ldots \text{(at Surface)}$$

Scanned by CamScanner

Inside sphere,
$\vec{E} = 0$.

$$\therefore \int_s^p dv = \int_s^p -E\,dr = \int_s^p 0\,dr = 0.$$

$$\therefore V_p - V_s = 0.$$

$$\therefore V_p = V_s$$

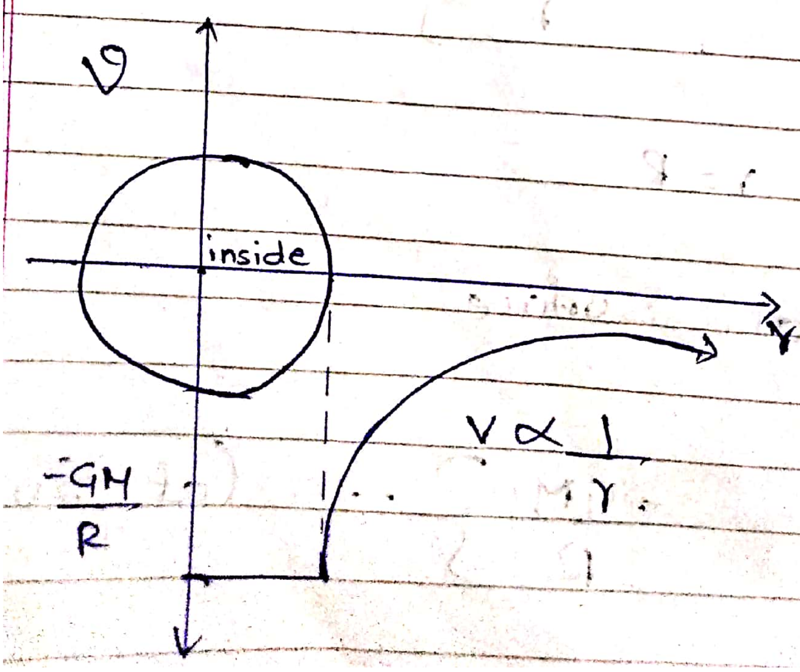
$$\therefore \left\{ V_i = V_p = V_s = -\frac{GM}{R} \right\}$$

$$\because E = -\frac{dv}{dr}, \quad 0 = -\frac{dv}{dr}.$$

$dv = 0 \quad \therefore V =$ constant.

$\therefore$ Potential from surface of Earth to inside and at center the Gravitational potential remains constant i.e $-GM/R$.

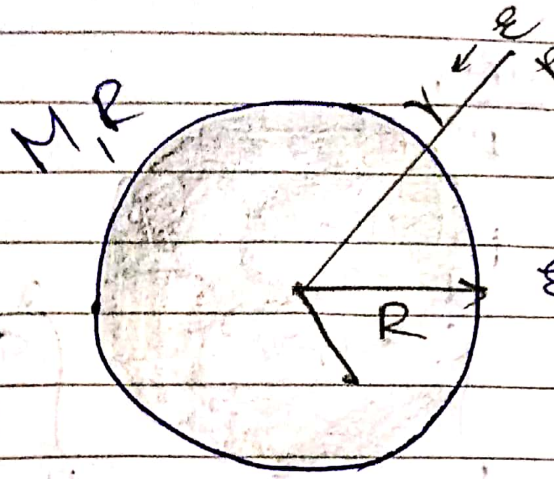
Graph:-



- $V_o = -\frac{GM}{r}$
- $\therefore V_o \propto \frac{1}{r}$
- $V_s = -\frac{GM}{R}$
- $V_i = -\frac{GM}{R}$

for Solid Sphere :-

$\mathcal{E} = -\frac{GM}{r^2}$



$\mathcal{E} = -\frac{GM}{R^2}$

• Outside,

$$\vec{\mathcal{E}} = \frac{-GM}{r^2}$$

potential $\therefore$ $$\int_{\infty}^{P} dV = \int_{\infty}^{P} -\mathcal{E}\,dr$$

$$\therefore V_P - V_\infty = \int_{\infty}^{r} \frac{GM}{r^2}\,dr.$$

$$\therefore V_P - V_\infty = -\frac{GM}{r^2}$$

$$\left\{ \therefore V_P = -\frac{GM}{r} \right\}$$

• At Surface,
the 'r' will be equal to R

potential $\therefore$ $$\left\{ V_s = -\frac{GM}{R} \right\}$$

- Inside Sphere,

$$\vec{E}_i = \frac{-GMr}{R^3}$$

& Gravitational potential,

$$= \int_s^P dv = \int_s^P -E\,dr = \int_R^r \frac{GMr\,dr}{R^3}$$

$$= V_P - V_S = \frac{GM}{R^3}\left[\frac{r^2}{2}\right]_R^r$$

$$\therefore V_P - V_S = \frac{GM}{R^3}\left[\frac{r^2}{2} - \frac{R^2}{2}\right]$$

$$\because V_S = -\frac{GM}{R}$$

$$\therefore V_P - \left(-\frac{GM}{R}\right) = \frac{GM}{R^3}\left[\frac{r^2 - R^2}{2}\right]$$

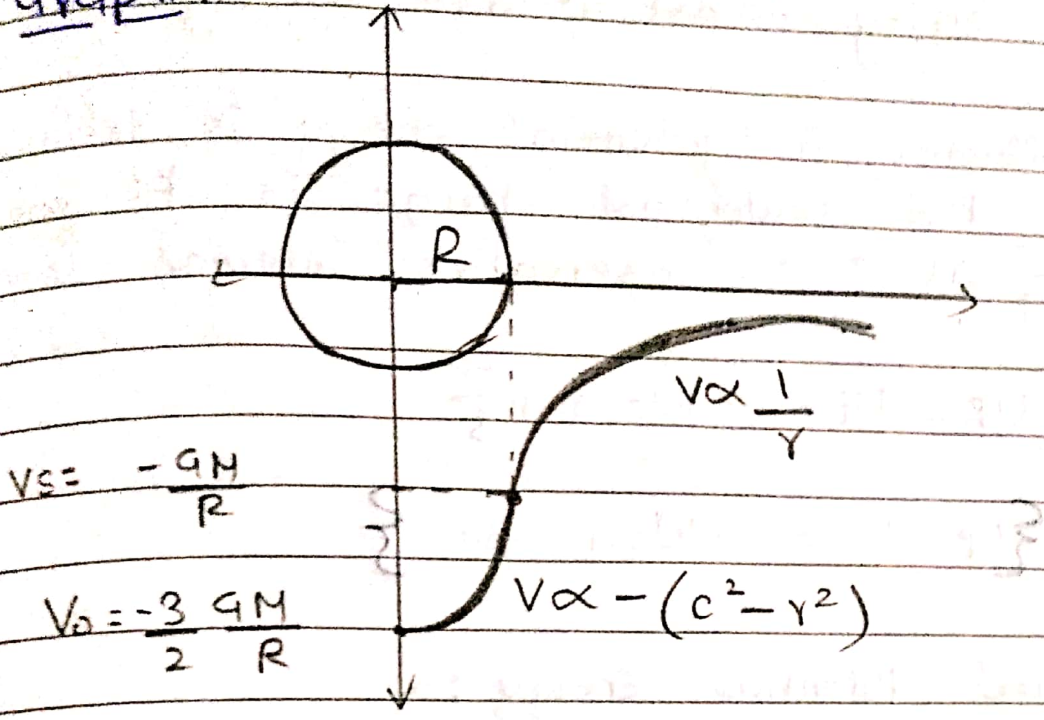
$$\therefore V_P = \frac{GM}{R^3}\left[\frac{r^2 - R^2}{2}\right] - \frac{GM}{R}$$

taking $\frac{GM}{R^3}$ common & multiplying by R^2,

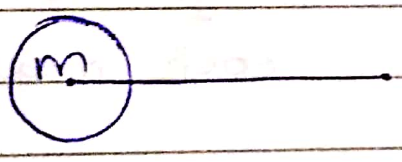
$$\therefore V_P = -\frac{GM}{R^3}\left[\frac{r^2 - R^2}{2}\right] + R^2$$

$$\left\{ \therefore V_P = \frac{GM}{R^3}\left[\frac{3R^2}{2} - \frac{1}{2}r^2\right] \right\}$$

Graph:-



k) Gravitational Potential Energy :- $(\vec{E})$:-

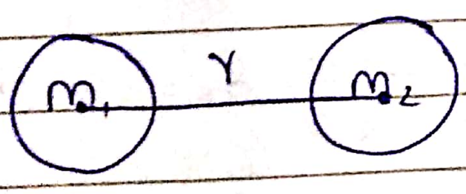


$$\vec{E} = \frac{-Gm}{R^2}$$

$$V = -\frac{GM}{r}$$

$$U = 0$$

Gravitational Potential Energy for a single mass is 0



here,
Gravitational Potential Energy

$$\left\{ = -\frac{Gm_1m_2}{r} \right\}$$

Potential Energy is not defined in Physics.

But difference in potential energy is defined provided K.E should not change i.e K.E = constant

i.e $\{\Delta U = -W_{\text{conservative internal force}}\}$

i.e $\{U_f - U_i = -W_{C.I.F}\}$

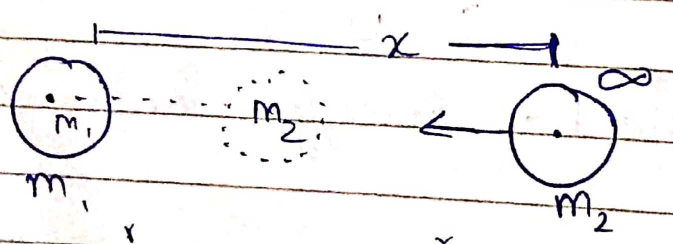
$= \{U_f - U_i = -W_{\text{grav. force}}\}$

Gravitational Potential Energy :-

At ∞ ; $U_\infty = 0$ (Assumed)

$= \{U_{config} - \cancel{U_\infty}^{\,0} = -W_{\text{grav. force}}$ in bringing each mass from $\infty\}$

★) Derivation :-



$, \therefore U_\infty = 0$

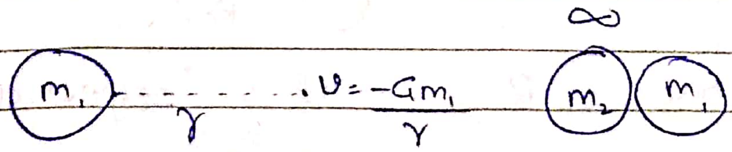
$$\therefore W = \int_\infty^r f \cdot dx = \int_\infty^r -\frac{Gm_1m_2}{x^2}dx = -Gm_1m_2\int_\infty^r \frac{dx}{x^2}$$

$$= Gm_1m_2\left[\frac{1}{x}\right]_\infty^r = \frac{Gm_1m_2}{r}.$$

$$\therefore \left\{U_{config} = -\frac{Gm_1m_2}{r}\right\}$$

$$\{\Delta U = +W_{ext}\}$$

*** Derivation without integration :-**

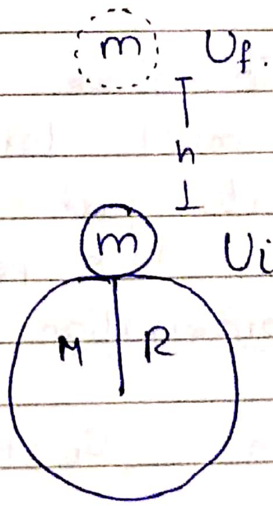


$$V = -\frac{Gm_1}{r}$$

$$W_2 = m\Delta V$$
$$= m_2(V_r - V_\infty)$$
$$= m_2\left(-\frac{Gm_1}{r} - 0\right)$$

$$\left\{W_2 = -\frac{Gm_1m_2}{r}\right\}$$

*) Potential Energy :-



We know $g = \frac{GM}{R^2}$

$$U_i = -\frac{GmM}{R}$$

$$U_f = -\frac{GmM}{(R+h)}$$

$$\therefore \Delta U = -\frac{GmM}{R} - \left(-\frac{GmM}{(R+h)}\right)$$
$$= -\frac{GmM}{R} + \frac{GmM}{R+h}$$
$$= -GMm\left(\frac{-1}{R} + \frac{1}{R+h}\right)$$
$$= -GMm\left(\frac{-\cancel{R} + \cancel{R} + h}{R(R+h)}\right) = \frac{-GmMh}{R(R+h)}$$

$$\therefore GM = gR^2$$

$$\left\{ \therefore \frac{gR^2 mh}{R(R+h)} = \frac{mgh}{\left(1+\frac{h}{R}\right)} \right\}$$

$\because h <<< R$, $\therefore$ it is neglected $\frac{h}{R} \to 0$

$$\therefore \left\{ \underset{U_H - U_S}{\Delta U} = mgh \right\}$$

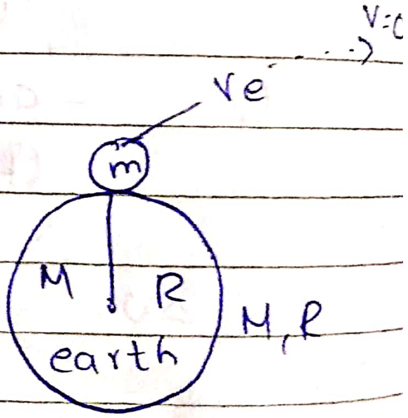
•) Escape Velocity :-

Earth's gravitational field end at ∞

To given a body it's escape velocity, we must throw it with a speed such that it just escape's earth gravitational field.
To escape $\to$ To ∞ (Total energy = 0)

•) Conservation of mechanical Energy :-

i.e $\quad U_i + k_i = U_f + k_f.$



$$= -\frac{GMm}{R} + \frac{1}{2} m V_e^2 = 0 + 0.$$

$$= \therefore \frac{1}{2} \cancel{m} V_e^2 = \frac{GM\cancel{m}}{R}$$

$$\left\{ \therefore V_e = \sqrt{\frac{2GM}{R}} \right\}$$

Valid only when body is escaping from surface.

•Valid for any planet

Escape Velocity for Earth :-

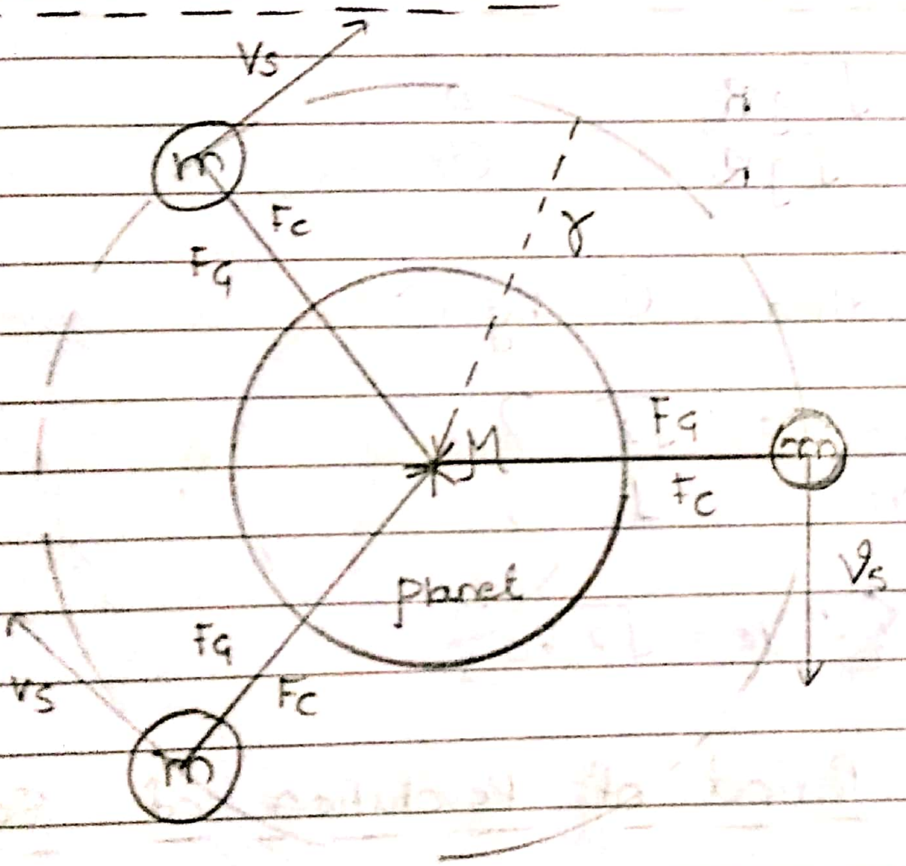
$$v_e = \sqrt{\frac{2GM}{R}}$$

$$v_e = \sqrt{\frac{2gR^2}{R}}$$

$$\left\{ v_e = \sqrt{2gR} \right\} = \{ 11.2 \text{ km/s} \}$$

•) Motion of Satellites :-

i.) Speed of Satellites :-



$$F_c = \frac{mv^2}{r} \quad , \quad F_G = \frac{GmM}{r^2}$$

$$\frac{\cancel{m}v^2}{\cancel{r}} = \frac{G\cancel{m}M}{r^{\cancel{2}}}$$

$$\left\{ v = \sqrt{\frac{GM}{r}} \right\} \rightarrow \text{Orbital Speed}$$

If Satellite is revolving very close to surface,

i.e $r \longrightarrow R$

$$\therefore V_o = \sqrt{\frac{GM}{R}} = \sqrt{\frac{gR^2}{R}}$$

$$\therefore \left\{ V_o = \sqrt{gR} \right\} \rightarrow \text{orbital Speed of Earth}$$

$$\left\{ = 7.9 \text{ km/sec} \quad \text{or} \quad 8 \text{ km/sec} \right\}$$

*) **Relationship b/w V_o And V_e :-**

$$V_e = \sqrt{2gR} \quad \cdots (1)$$
$$V_o = \sqrt{gR} \quad \cdots (2)$$

Dividing (1) by (2),

$$\left\{ \frac{V_e}{V_o} = \frac{\sqrt{2}}{1} \right\}$$

$$\left\{ \therefore V_e = \sqrt{2}\, V_o \right\}$$

*) **Time Period of Revolution of Satellite :-**

$$\text{Time period} = \frac{\text{Distance}}{\text{Speed}} = \frac{2\pi r}{v}$$

$$\left\{ = \frac{2\pi r^1}{\sqrt{\frac{GM}{r}}} = T = \frac{2\pi r^{1+\frac{1}{2}}}{\sqrt{GM}} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}} \right\}$$

$$\left\{ T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{gR^2}} = T = K r^{\frac{3}{2}} \right\}$$

$$\left\{ \text{i.e} \quad T \propto r^{\frac{3}{2}} = T^2 \propto r^3 \right\}$$

★ Time Period of Revolution of Satellite close to Earth:-

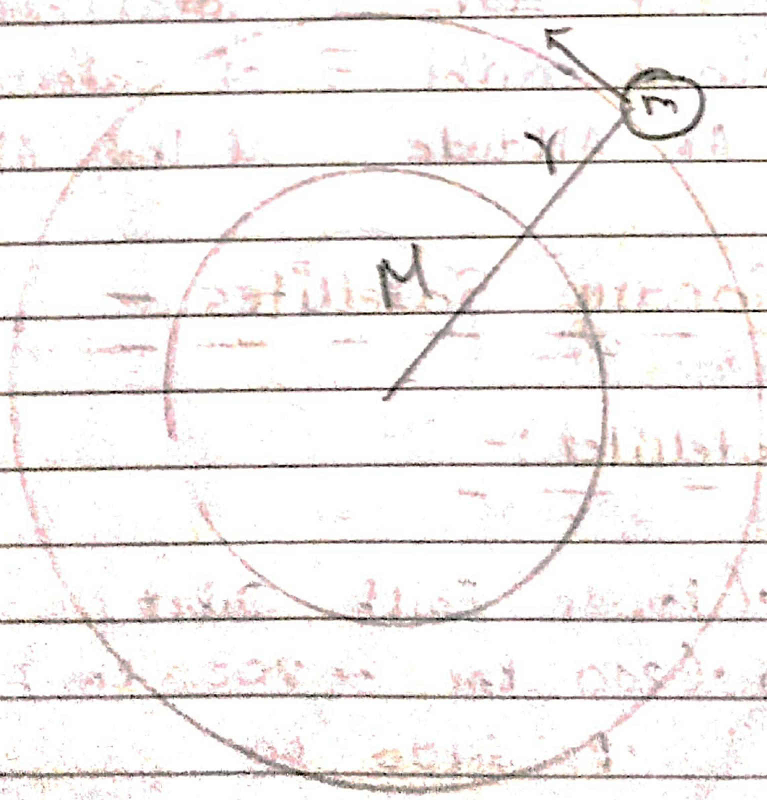
$r \longrightarrow R$

$$\therefore T = \frac{2\pi r}{v} = \frac{2\pi R}{\sqrt{\frac{GM}{R}}}$$

$$\left\{ T = \frac{2\pi R}{\sqrt{\frac{gR^2}{R}}} = \frac{2\pi R}{\sqrt{gr}} = 2\pi\sqrt{\frac{R}{g}} \right\}$$

comes out to be 84.6 min.

***) Energy of a Satellite :-**



If TE = -ve (Bounded System) (Circle/ellipse)
If TE = 0, (free System) (Parabola)
If TE = +ve (free System) (Hyperbola)

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$K.E = \frac{1}{2}mv^2$

$= \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \frac{1}{2}\frac{mGM}{r}$

$$\left\{ \therefore KE = \frac{GMm}{2r} \right\} \quad \text{--- (1)}$$

and potential Energy,

$$\left\{ U = \frac{-GmM}{r} \right\} \quad \text{--- (2)}$$

$\therefore$ Total energy = (1) + (2) = KE + U

$$\left\{ \therefore TE = \frac{-GMm}{2r} \right\} \quad \{\text{Binding Energy}\}$$

Total Energy is negative becoz satellite revolve in a bounded Curve.

{ Total Energy of Star Satellite above earth Surface at An Altitude = Total Energy of Satellite at that Altitude − Total Energy at Surface }

* Geo-Stationary Satellites —

1.) Types of Satellites :-

a.) L.E.O — (lower Earth Orbit)

• 1 day = 12 revo. (200 km ~ 2000 km)

• R = 6400 km

• T = 1.5 ~ 2 hrs

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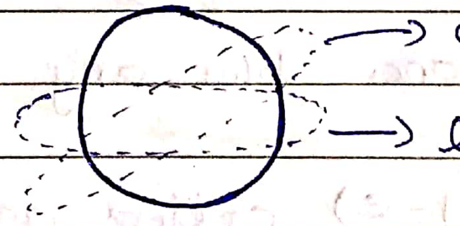
b.) M.E.O – (Middle Earth Orbit)
- (20,000 km) , (2000 km ~ 36,000 km)
- T = 12 hr.
- 1 day = 2 revolutions

c.) G.E.O – (Geostationary orbit)
- (36,000 km)
- T = 24 hr
- 1 day = 1 revolution.

*.) Geostationary Satellites :-

- 1 revolution in 24 hrs.
- Which revolves around such earth such that it appears stationary from any point of Earth.
- Which has same sense of rotation as that of Earth
- Which is placed at an altitude of 36,000 km above earth.
- Geostationary Statellites revolves around equator belt
- Geosynchronous Satellite (T = 24 hr, h = 36,000 km), but inclined.



- Velocity of G.E.S – $F_G = F_C$

$$= \frac{GmM}{r^2} = \frac{mv^2}{r}$$

$$\therefore v = \sqrt{\frac{GM}{r}}$$

- Time period of satellites :-

$$= \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$$

$$\left\{ T = \frac{2\pi r^{1+\frac{1}{2}}}{\sqrt{GM}} = \frac{2\pi r^{3/2}}{\sqrt{gR^2}} \right\}$$

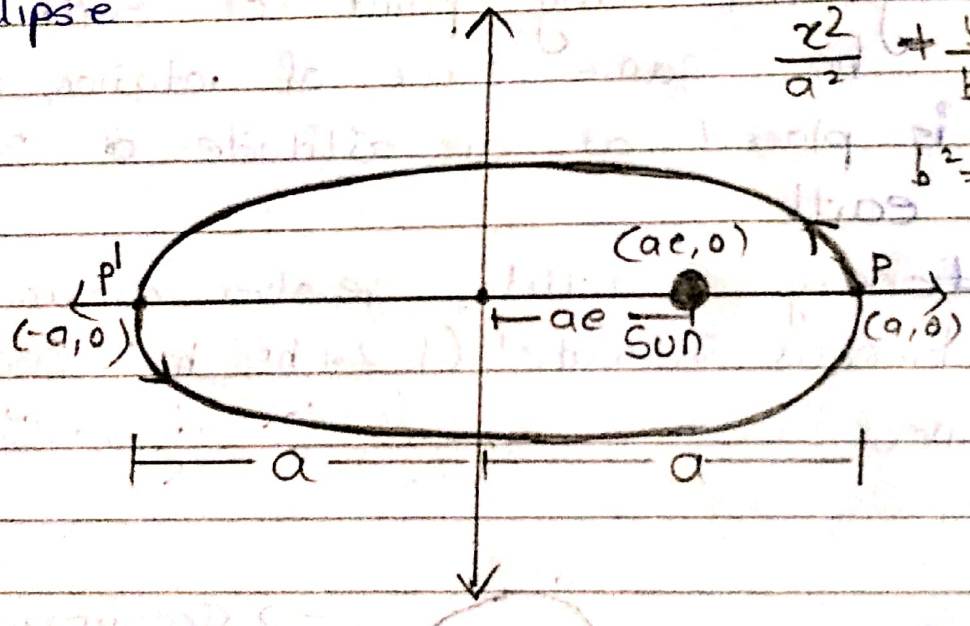
$$\left\{ \text{i.e. } T = k r^{3/2}, \quad T^2 \propto r^3 \right\}$$

*) kepler's law of planetary Motion :-

1.) First law - All planets revolve around sun in closed elliptical orbits & sun is at one focus of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$b^2 = a^2(1-e^2)$$



The closest distance b/w any planet & sun is,

$r = a - ae$

or $r = a(1-e)$ called as 'Periphelion or Perigee'

The farthest distance b/w sun & planet is,

$r = a + ae$

$r = a(1+e)$ called as 'Aphelion' or 'Aphogee'

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2.) Second law - Planets do not revolve around sun with constant speeds.

i.e Planets move faster when it is near to sun and slower when it is farthest from sun.

'Actual law' :-

'The Areal velocity of a planet is constant.'

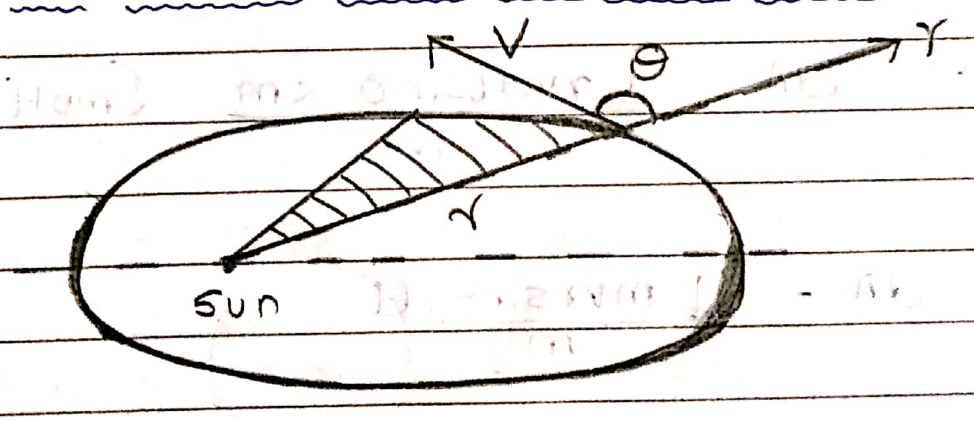
or

'A planet sweeps equal area in equal time around sun.

i.e $\frac{dA}{dt}$ = constant

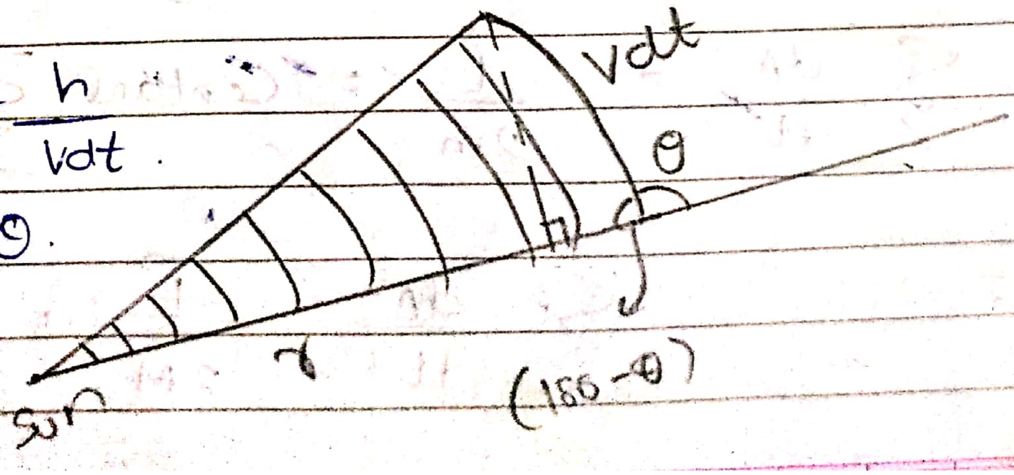
or $\frac{A_1}{t_1} = \frac{A_2}{t_2} = \frac{A_3}{t_3}$ = constant.

*) Proof for kepler's Second law :-



$\sin(180-\theta) = \frac{h}{vdt}$.

$\therefore h = vdt \sin\theta$.



Planet revolving around Sun has constant Angular momentum.

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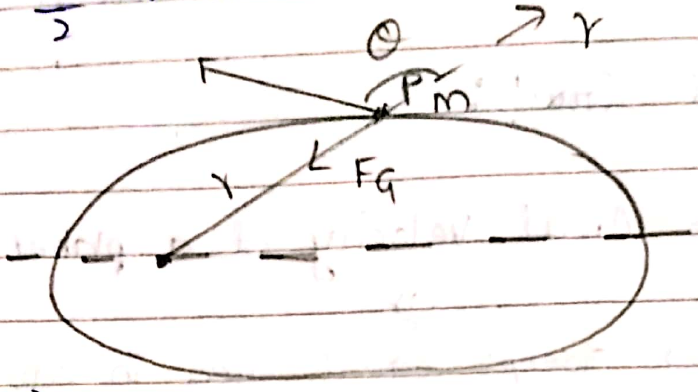
Planet moves for time 'dt'

$\therefore$ Distance covered by Planet = Vdt

Area = $\frac{1}{2} \times b \times h$

$$dA = \frac{1}{2} \times r \times Vdt\sin\theta \quad \ldots (1)$$



$\tau = \vec{r} \times \vec{F} = 0$

$= rF\sin 180 = 0.$

if $\tau_{net} = 0$

then $\vec{L}$ = Constant.

i.e $\vec{L} = \vec{r} \times \vec{P}$

$= \vec{r} \times m\vec{V}$

$L = mVr\sin\theta$ = constant $\ldots (2)$

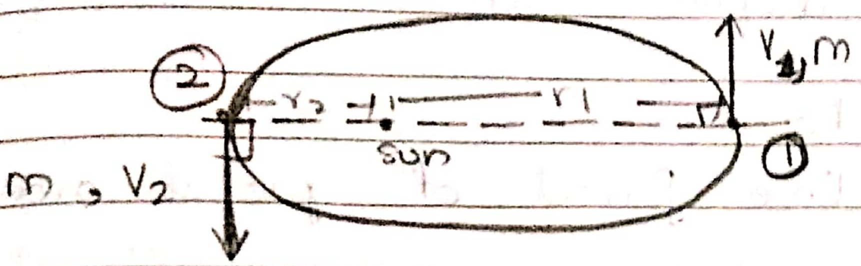
$\because dA = \frac{1}{2} \frac{rVdt\sin\theta \times m}{m}$ (multiplying eq^n (1) by $\frac{}{m}$)

$$dA = \frac{1}{2} \frac{mVr\sin\theta\, dt}{m}.$$

$$\left\{ \frac{dA}{dt} = \frac{\vec{L}}{2m} = \text{Constant} \right\}$$

$$\therefore \left\{ \frac{dA}{dt} = \frac{\vec{L}}{2M} = \text{Constant} \right\}$$

*) Speed of Planets at aphogee & Perigee :-



∵ Angular momentum is 'C', i.e $\vec{\tau} = 0$, becos $\tau = 0$, (about sun),

then, $L_1 = L_2$

$$r_1 \times P_1 = r_2 \times P_2$$

$$r_1 \times m\vec{v_1} = r_2 \times m\vec{v_2} \qquad r_1 = a(1+e)$$

$$v_1 r_1 = v_2 r_2 \qquad r_2 = a(1-e)$$

$$\therefore v_1 a(1+e) = v_2 a(1-e) \quad \ldots \quad (1)$$

Applying Conservation of energy at pt (1) & (2),

i.e $T.E_1 = T.E_2$

$$= U_1 + k_1 = U_2 + k_2$$

$$= \frac{-GM_s m}{a(1+e)} + \frac{1}{2} m v_1^2 = -\frac{GM_s m}{a(1-e)} + \frac{1}{2} m v_2^2$$

$$\ldots \quad (2)$$

on solving eq^n (1) & (2),

$$\underset{(\text{Aphogee})}{V_1} = \sqrt{\frac{GM_s}{a}\left(\frac{1-e}{1+e}\right)} \quad \& \quad \underset{(\text{Perigee})}{V_2} = \sqrt{\frac{GM_s}{a}\left(\frac{1+e}{1-e}\right)}$$

) Third law :-

let us assume planet moves in a circular orbit around sun

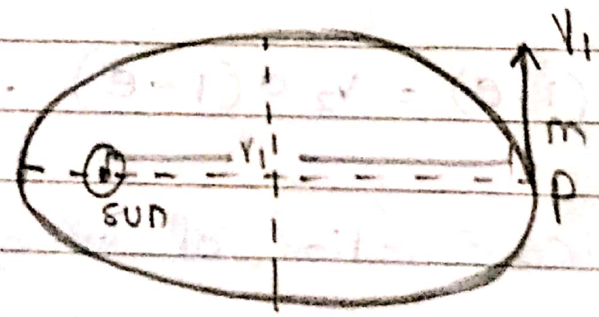
$\therefore F_C = F_G$.

Then time Period of planet around sun

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM_s}}$$

$$T \propto r^{\frac{3}{2}}$$

Actual Proof :- $(T^2 \propto a^3)$



We know, $\frac{dA}{dt} = \frac{L}{2m}$

1 complete revolution

: Area Swept = Area of ellipse $= \pi ab$

time $= T$.

$$\therefore \frac{\pi ab}{T} = \frac{L}{2m}$$

$$= \frac{\pi ab}{T} = \frac{r_1 m v_1}{2m} \quad (1)$$

Angular momentum = $\vec{r} \times p = r_1 m v_1$

$\therefore r_1 = a + ae = a(1+e)$

$$\therefore v_1 = \sqrt{\frac{GM_s}{a}\left(\frac{1-e}{1+e}\right)}$$

Putting this value is eq^n (1), $\because$ Angular momentum is constant.

$$\therefore \frac{\pi \cancel{a} b}{T} = \cancel{a}(1+e)\sqrt{\frac{GM_s}{a}\left(\frac{1-e}{1+e}\right)}$$

Squaring both sides,

$$\frac{\pi^2 b^2}{T^2} = \frac{(1+e)^2 GM_s (1-e)}{a \quad (1+e)} \qquad [\because b^2 = a^2(1-e^2)]$$

$$= \frac{\pi^2 a^3 \cancel{(1-e^2)}}{T^2} = \cancel{(1+e)} GM_s \cancel{(1-e)}$$

$$\therefore T^2 = \frac{\pi^2 a^3}{GM_s}$$

$$\left\{ \therefore T^2 \propto a^3 \right\}$$

Hence proved